Pergamon

Int. J. Heat Mass Transfer. Vol. 39, No. 17, pp. 3761–3764, 1996

0017–9310/96 $15.00+0.00

PII: S0017–9310(96)00049–X

TECHNICAL NOTE

Use of two-dimensional Fast Fourier Transform in harmonic modulated thermal diffusion

FRANÇOISE DEPASSE and PHILIPPE GROSSEL

Groupe de Recherche Surfaces et Matériaux, Laboratoire d'Energétique et d'Optique, Faculté des Sciences de Reims, BP 1039, 51 687 Reims Cedex 2, France

(*Received* 14 *March* 1995 *and in final form* 4 *January* 1996)

INTRODUCTION

Numerous nondestructive evaluation experiments use thermo-physical characterization of surface or sub-surface profiles by means of harmonic modulated heat source methods.

In this case, the heat diffusion equation is written as the thermal equivalent of the Helmholtz equation, which can be solved with the aid of different integral transforms [1–3]. Among these, one presents great interest for problems with translational symmetry: the plane wave spectrum (PWS) [4, 5]. This method is often used in near-field optics and can be applied to solving all differential equations with the appearance of the Helmholtz equation—quantum mechanical, acoustic [6] or thermal areas [7, 8].

In this note, the principal theoretical aspects of the PWS are collected so as to be used in the thermal area. The method is simple and gives the a.c. temperature field due to the heat plane source. The three-dimensional diffusion equation is solved with the aid of the two-dimensional Fourier transform. Straightforward analytical calculations give connections with the axisymetrical (Hankel transform) and point source problems. The analytical expressions lead to the direct use of Fast Fourier Transform (FFT) algorithms available in computer mathematical software. Simple application of an example, via the FFT algorithms, shows the ease and interest of this general method.

THEORETICAL BACKGROUND

In the case of a homogeneous and isotropic medium, the differential equation verified by the thermal diffusion is

$$\Delta T(\mathbf{R},t)-\frac{1}{\alpha}\frac{\partial T(\mathbf{R},t)}{\partial t}=-G(\mathbf{R},t) \qquad (1)$$

where the localized source term G and the thermo-physical properties of the medium: diffusivity α, thermal conductivity κ, mass density ρ and specific heat c are defined

$$G(\mathbf{R},t)=\frac{S(\mathbf{R},t)}{\kappa} \quad \alpha=\frac{\kappa}{\rho c} \qquad (2a,b)$$

S is the deposited power at location $\mathbf{R}$ and time t.

The theory here is restricted to the case of one single modulation frequency ω for the source S, as it is currently used in photothermal experiments [7]. We can introduce the complex harmonic solution of equation (1) as

$$G(\mathbf{R},t)=G(\mathbf{R})\,\mathrm{e}^{-i\omega t} \quad T(\mathbf{R},t)=T(\mathbf{R})\,\mathrm{e}^{-i\omega t} \qquad (3a,b)$$

giving the time-independent equation, i.e. the thermal Helmholtz equation [6]

$$\Delta T(\mathbf{R})+\frac{i\omega}{\alpha}T(\mathbf{R})=-G(\mathbf{R}). \qquad (4)$$

The three-dimensional Fourier transform of the $T(\mathbf{R})$ fields is defined as

$$\tilde{T}(\mathbf{k})=\tilde{T}(k_x,k_y,k_z)=\frac{1}{(2\pi)^{3/2}}\iiint \mathrm{d}\mathbf{R}\,\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{R}}T(\mathbf{R}) \qquad (5)$$

and a similar expression is obtained for the $G(\mathbf{R})$ field. Then the integral transform of equation (4) gives

$$-k^2\tilde{T}(\mathbf{k})+\frac{i\omega}{\alpha}\tilde{T}(\mathbf{k})=-\tilde{G}(\mathbf{k}). \qquad (6)$$

Solving equation (6) for $\tilde{T}(\mathbf{k})$ and inverting the result gives the temperature field as

$$T(\mathbf{R})=\frac{-1}{(2\pi)^{2/3}}\iiint \mathrm{d}\mathbf{k}\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{R}}\frac{\tilde{G}(\mathbf{k})}{i\omega/\alpha-k^2} \qquad (7)$$

with

$$k^2=k_x^2+k_y^2+k_z^2. \qquad (8)$$

The following will be particularly concerned with plane sources; but the temperature response to such sources could be totalled (in the linear thermophysical area) for any three-dimensional sources. With the simple plane geometry, it is interesting to differentiate the plane (y,z) of the source from the perpendicular axis (x). The vectors $\mathbf{R}$ and $\mathbf{k}$ can be rewritten as

$$\mathbf{R}=(x,\mathbf{r})=(x,y,z) \quad \mathbf{k}=(k_x,\mathbf{q})=(k_x,k_y,k_z). \qquad (9a,b)$$

The direct and inverse source terms are specified as

$$G(\mathbf{R})=G(\mathbf{r},x)=A(\mathbf{r})\,\delta(x-x_e)$$

$$\tilde{G}(\mathbf{k})=\mathbb{A}(k_y,k_z)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-ik_xx_e}=\mathbb{A}(\mathbf{q})\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-ik_xx_e}, \qquad (10a,b)$$

where $\mathbb{A}(\mathbf{q})$ is the two-dimensional Fourier transform of the plane source $A(\mathbf{r})$. Equation (10b) is introduced into equation (7) to obtain

$$T(\mathbf{R})=-\frac{1}{2\pi}\iint \mathrm{d}\mathbf{q}\,\mathbb{A}(\mathbf{q})\,\mathrm{e}^{i\mathbf{q}\cdot\mathbf{r}}\left\{\frac{1}{2\pi}\int \mathrm{d}k_x\frac{\mathrm{e}^{ik_x(x-x_e)}}{\chi(\mathbf{q})^2-k_x^2}\right\} \qquad (11)$$

NOMENCLATURE

A	two-dimensional source term
c	specific heat
G	source term in thermal diffusion equation
G	three-dimensional Fourier space source term
$\mathbb{G}$	two-dimensional Fourier space source term
J_0	Bessel function of first kind of order zero
$\mathbf{k}$	three-dimensional Fourier space vector
$\tilde{\mathbf{k}}$	one-dimensional complex thermal wave-vector
p	heat power in the point source case
$\mathbf{q}$	two-dimensional Fourier space vector
$\mathbf{r}$	two-dimensional location vector
$\mathbf{R}$	three-dimensional location space vector
$\mathscr{R}$	reflection coefficient of the a.c. thermal solution from interface
S	heat power source term
T	complex a.c. temperature field
$\tilde{T}$	three-dimensional Fourier transform of a.c. temperature field
$\mathbb{T}$	two-dimensional Fourier transform of a.c. temperature field.

Greek symbols

α	diffusivity
ρ	mass density
ω	angular frequency
κ	thermal conductivity
λ	radial coordinate in the two-dimensional Fourier space
χ	transfer term of the plane wave spectrum.

with

$$\chi(\mathbf{q}) = \sqrt{\frac{i\omega}{\alpha} - k_1^2 - k_2^2}. \quad (12)$$

The inner integration (11) can easily be performed in the complex plane. It finally gives

$$T(\mathbf{R}) = \frac{i}{4\pi}\iint \mathrm{d}\mathbf{q}\, \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\mathbb{A}(\mathbf{q})\frac{\exp(i\chi|x-x_e|)}{\chi};$$

$$\mathrm{Im}(\chi) \geqslant 0; \quad \mathrm{Re}(\chi) \geqslant 0. \quad (13)$$

Equation (13) determines completely the temperature field at any point $\mathbf{R}$, from the data of the plane source $G(\mathbf{R})$.

The (two-dimensional) PWS of the temperature created by the plane source is thus

$$\mathbb{T}(0,\mathbf{q}) = \frac{i}{2\chi}\mathbb{A}(\mathbf{q}). \quad (14)$$

More generally, without source, the transfer function from a known temperature field located at the plane $x = 0$ to the plane $x(>0)$ can be written as

$$T(\mathbf{R}) = T(x,\mathbf{r}) = \frac{1}{2\pi}\iint \mathrm{d}\mathbf{q}\, \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}\mathbb{T}(0,\mathbf{q})\exp(\mathrm{i}\chi x) \quad (15)$$

with

$$\mathbb{T}(0,\mathbf{q}) = \frac{1}{2\pi}\iint \mathrm{d}\mathbf{r}\, \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\mathbf{r}}T(0,\mathbf{r}). \quad (16)$$

As formulas (13) and (15) show, the theory can be directly used by FFT algorithms.

The PWS method is an exact solution of the thermal equation (4) and can be linked to the usual solutions. In the case of an axisymmetrical source with its centre located at $(x_e, 0, 0)$, the general two-dimensional transform can be modified to

$$T(\mathbf{R}) = \frac{i}{2}\int_0^{\infty} \mathrm{d}\lambda\lambda\mathbb{A}(\lambda)\frac{\exp[i|x-x_e|\sqrt{i\omega/\alpha-\lambda^2}]}{\sqrt{i\omega/\alpha-\lambda^2}} \times \int_0^{2\pi} \mathrm{d}\theta\exp[i\lambda r\cos(\theta)]. \quad (17)$$

This gives the temperature field from the zeroth Hankel transform $\mathbb{A}(\lambda)$ of the source term $A(r)$

$$\mathbb{A}(\lambda) = \int_0^{\infty} \mathrm{d}r r J_0(r\lambda)A(r). \quad (18)$$

Thus

$$T(\mathbf{R}) = \frac{i}{2}\int_0^{\infty} \mathrm{d}\lambda\lambda J_0(\lambda r)\mathbb{A}(\lambda)\frac{\exp[i|x-x_e|\chi(\lambda)]}{\chi(\lambda)}. \quad (19)$$

Such a field $T(\mathbf{R})$ can be totalled on different plane sources in order to give the total response for a three-dimensional heat source.

Another interesting case is the point source of strength p

$$G(\mathbf{R}) = \frac{p}{\kappa}\delta(x-x_e)\,\delta(y-y_e)\,\delta(z-z_e). \quad (20)$$

In using the Weyl's formula [5]

$$\frac{\exp(i\tilde{\mathbf{k}}|\mathbf{R}-\mathbf{R}_e|)}{|\mathbf{R}-\mathbf{R}_e|} = \frac{\mathrm{i}}{2\pi}\iint \mathrm{d}\mathbf{q}\, \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}_e)}\frac{\exp(i\chi|x-x_e|)}{\chi}. \quad (21)$$

This leads to

$$T(\mathbf{R}) = \frac{p}{4\pi\kappa}\frac{\exp(i\tilde{\mathbf{k}}|\mathbf{R}-\mathbf{R}_e|)}{|\mathbf{R}-\mathbf{R}_e|} \quad (22)$$

with

$$\tilde{\mathbf{k}} = \sqrt{\frac{i\omega}{\alpha}} \quad (23)$$

the complex thermal wave vector appearing in general one-dimensional problems.

The resulting equation (22) corresponds, of course, to the Green's function for equation (4). This enlightens the characteristics of the PWS. The decrease of the magnitude of the temperature field [equation (13)] is entirely described by the transfer function in the x-direction. This is a common feature with the near field of optical source, where the homogeneous waves as well as the evanescent waves can be written solely by using PWS [4, 5].

EXAMPLE

The chosen example corresponds to the theoretical analysis of the recently observed thermal-wave interferences [9]. An aluminium layer (thermal conductivity $\kappa_1 = 200$ W m^{-1} K^{-1}, diffusivity $\alpha_1 = 8.5\ 10^{-5}$ m^2 s^{-1}, thickness $L = 1$ mm) is covered with a thin layer of black paint which permits total surface absorption. Two in-phase modulated Gaussian laser beams (power 10 mW, radius 300 μm, modulation frequency 18 Hz) hit the surface and generate in the metal layer the a.c. component of temperature field $T(\mathbf{R})$, submitted to the Fourier type boundary conditions [2, 8] at the air–Al interface, with a convective heat transfer coefficient H_e ($H_e = 10$ W m^{-2} K^{-1}).

$$-\kappa_1 \frac{\partial}{\partial x} T(0,\mathbf{r})_{x=0} = A(\mathbf{r}) - H_e T(0,\mathbf{r}). \quad (24)$$

The rear medium is constituted by a semi-space with a thermal conductivity κ_2 and diffusivity α_2; so the continuity equations at the interface are written as

$$-\kappa_1 \frac{\partial}{\partial x} T_{1_{x=L}} = -\kappa_2 \frac{\partial}{\partial x} T_{2_{x=L}} \quad T_1(L,\mathbf{r}) = T_2(L,\mathbf{r}). \quad (25a,b)$$

This simple geometry can be analysed in terms of the PWS by using the Fourier transform of the Helmholtz equation (4) and of the continuity equations (24) and (25). This gives without any approximation

$$T(\mathbf{R}) = T(x,\mathbf{r}) = \frac{1}{2\pi}\iint d\mathbf{q}\, e^{i\mathbf{q}\cdot\mathbf{r}}\,\mathbb{T}(0,\mathbf{q}) \times [\exp(i\chi_1 x) + \mathscr{R}(\mathbf{q})\exp(-i\chi_1 x)], \quad (26)$$

where the PWS for this problem takes the source, the convective coefficient and the geometry into account.

$$\mathbb{T}(0,\mathbf{q}) = -\frac{\mathbb{A}(\mathbf{q})}{[i\kappa_1\chi_1(\mathbf{q})\{1-\mathscr{R}(\mathbf{q})\} - H_e\{1+\mathscr{R}(\mathbf{q})\}]} \quad (27)$$

in which $\mathscr{R}(\mathbf{q})$ is a reflection coefficient, from the rear interface [6].

$$\mathscr{R}(\mathbf{q}) = \frac{\kappa_1\chi_1(\mathbf{q}) - \kappa_2\chi_2(\mathbf{q})}{\kappa_1\chi_1(\mathbf{q}) + \kappa_2\chi_2(\mathbf{q})} \exp\{2i\chi_1(\mathbf{q})L\}. \quad (28)$$

Figure 1 gives the amplitudes of the a.c. temperature field on both sides of the Al layer (front and rear). Figure 2 shows details of the amplitude and the phase of the a.c. temperature field obtained with formulas (27) and (28) at the rear interface. The thermo-physical properties of the second material are that of polyvinylidene fluoride film ($\kappa_2 = 0.13$ W m^{-1} K^{-1}, $\alpha_2 = 5.4\ 10^{-8}$ m^2 s^{-1}). The function of the convective coefficient rapidly becomes negligible with increasing modulation frequency. The calculations have been performed with a MATLAB^R two-dimensional FFT algorithm; the result is obtained within some seconds with a 128×128 grid. The diffusion of the temperature field appears clearly on Figs. 1 and 2.

CONCLUSION

In this note, some formal and computing aspects of the plane wave spectrum expansion have been gathered. This powerful theoretical method, generally used in optics, particularly in near-field optics, is well adapted to the solution of the a.c. thermal diffusion equation with sources in the case of a homogeneous medium. This particular form of integral transform is specially adapted to plane boundary conditions. The obtained expressions are directly transposed in two-dimensional FFT algorithms.

A generalization of a multilayer matrix [7] or quadripole [10] algorithm can be directly formulated and can give a.c. thermal responses for any shape source.

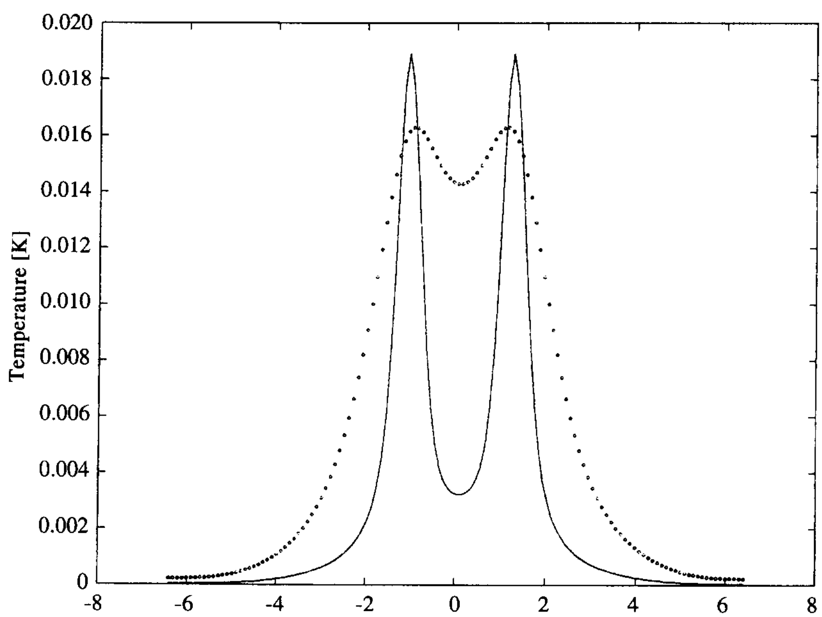

Fig. 1. Magnitude of the 18 Hz a.c. temperature field in front (- - -) and at the rear (○ ○ ○ ○) interfaces of the 1 mm thick 300 μm radius. The rear amplitude is augmented by a factor 5.

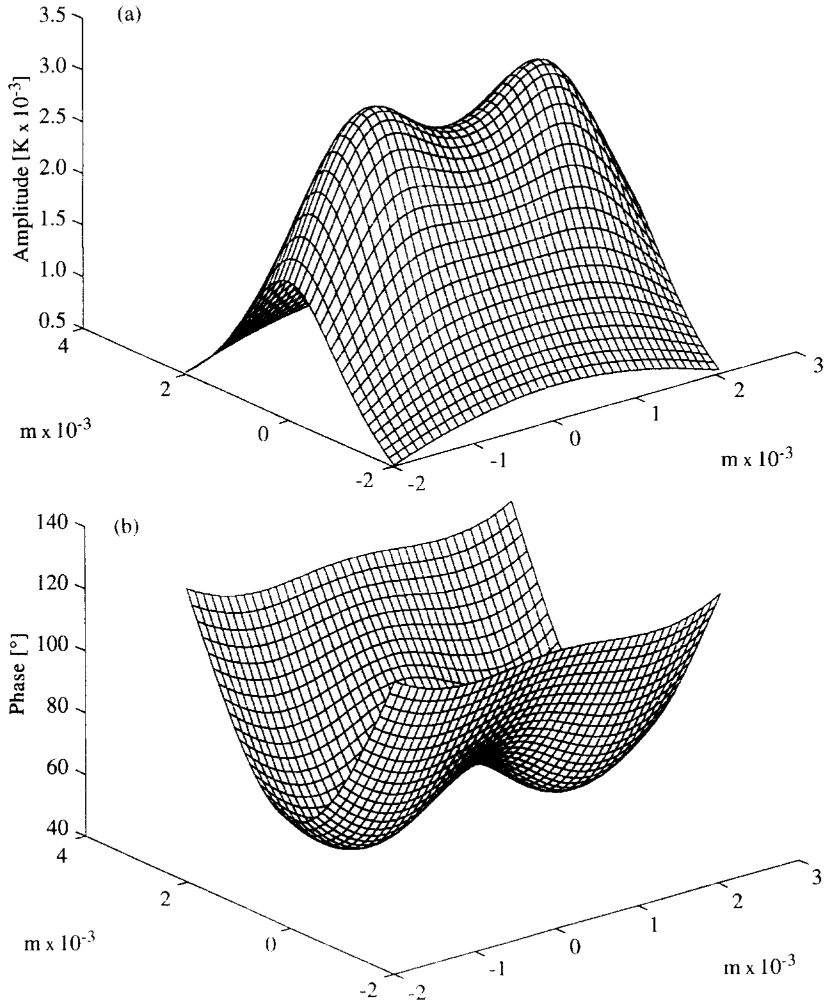

Fig. 2. (a) Part of the magnitude map of the a.c. temperature field in the case of two in-phase Gaussian sources. The distance between the two sources is 2.4 mm and the studied depth 1 mm. (b) Part of the phase lag map of the a.c. temperature field. The phase reference is that of the modulated source signal.

REFERENCES

1. A. Mandelis, Theory of photothermal-wave diffraction and interference in condensed media. *J. Opt. Soc. Am. A* **6**, 298–308 (1989).
2. N. Özisik, *Heat Conduction*. Wiley, New York (1990).
3. A. Mandelis, Hamilton–Jacobi formulation and quantum theory of thermal wave propagation in solid state, *J. Math. Phys.* **26**, 2676–2683 (1985).
4. M. Nieto-Vesperinas, *Scattering and Diffraction in Physical Optics*. Wiley, New York (1993).
5. J. A. Stratton, *Electromagnetic Theory*. McGraw-Hill, New York (1941).
6. M. Nikoonahad and E. A. Ash, Ultrasonic focusing in absorptive fluids. In *Acoustical Imaging* (Edited by E. A. Ash and C. R. Hill), Vol. 12, p. 47. Plenum, New York (1982).
7. M. Vaez Iravani and H. K. Wickramasinghe, Scattering matrix approach to thermal wave propagation in layered structures, *J. Appl. Phys.* **58**, 122–131 (1985).
8. N. Özisik, *Boundary Values Problems of Heat Conduction*. Dover, New York (1989).
9. A. Mandelis and K. F. Leung, Photothermal-wave diffraction and interference in condensed media: experimental evidence in aluminium, *J. Opt. Soc. Am.* **8**, 186–200 (1991).
10. J. C. Batsale, D. Maillet and A. Degiovanni, Extension de la méthode des quadripôles thermiques . . . défaut plan bidimensionnel, *Int. J. Heat Mass Transfer* **37**, 111–127 (1994).